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SHORTER COMMUNICATION

NUMERICAL SOLUTION OF THE EMBEDDING EQUATIONS APPLIED TO THE LANDAU MELTING PROBLEM

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NOMENCLATURE

A, integral function;
a, terminal recession distance intercept;
c, specific heat [J/kg K];
F, dimensionless fictitious heat flux;
G, modified fictitious heat flux;
H, applied heat flux [W/m^2];
$\tilde{H}$, fictitious heat flux [W/m^2];
I, integral function;
k, thermal conductivity [W/m K];
L, latent heat of fusion [J/kg];
m, Stefan number;
N, subscript denoting current time level;
s, recession distance [m];
$\dot{s}$, recession velocity (ds/dt) [m/s];
T, temperature [K];
T_0, initial slab temperature [K];
T_m, melt temperature [K];
t, time [s];
t_m, melt time [s];
v, terminal recession velocity [m/s];
x, inertial spatial coordinate [m];
y, dimensionless time $(t-t_m)/t_m$.

Greek symbols
α, thermal diffusivity [m^2/s];
ζ, dimensionless inertial coordinate;
θ, dimensionless temperature $(T-T_0)/(T_m-T_0)$;
ν, dimensionless terminal recession velocity;
ξ, dimensionless recession distance;
ξ', dimensionless recession velocity ($d\xi/dy$);
π, 3.14159265358979323846264 3;
ρ, density [kg/m^3].

THE PURPOSE of this short communication is to report briefly the algorithm and some of the results of a numerical solution of Boley's [1] embedding equations applied to the Landau melting problem (LMP) [2] shown in Fig. 1. Except for the case $L = \infty$, which is linear for constant thermal properties, there are no known exact solutions to this generally nonlinear problem for times $t > t_m$. As $t \to \infty$, the LMP does have a known steady state solution in $x - s(t)$ [2] and $\dot{s}(t)$ has the terminal value v:

$$v = \frac{H}{\rho[L + c(T_m - T_0)]}. \qquad (1)$$

Details of both the numerical algorithm and results are given in [3], an evaluation of several finite difference techniques of ablation analysis applied to the LMP using the numerical solution of Boley's equations as a standard of comparison.

Boley [1] proposed embedding the LMP in an inverse heat conduction problem with no moving boundaries by applying a fictitious heat flux $H + \tilde{H}(t)$ to the surface $x = 0$ and assuming the existence of a fictitious solid with the same thermal properties as the unablated solid in the region $0 \leqslant x < s(t)$ after melting starts. With the scaling of physical variables shown in Table 1, one can show that solutions of the LMP change only with a Stefan number $m = \pi^{1/2}c(T_m - T_0)/(2L)$. The problem is to find $F(y)$ and $\xi(y)$ given the physical boundary conditions for $t > t_m$ in Fig. 1. For $m < \infty$, $\xi'(0) = 0$ but for $m = \infty$, [2] shows that $\xi'(0) = 1/(2\pi^{1/2})$. In all cases $F(0) = \xi(0) = 0$.

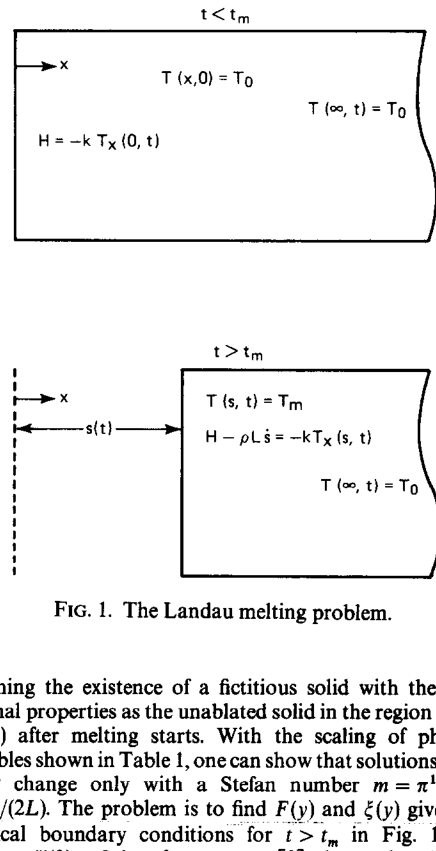

FIG. 1. The Landau melting problem.

Boley [1] obtained general nonlinear integrodifferential equations for $\xi(y)$ and $F(y)$ for $0 < m < \infty$ and was able to obtain series solutions for $0 < y \ll 1$. He proposed a numeri-

Table 1. Definition of dimensionless variables

Physical variable	Dimensionless variable	Definition
x	ζ	$x/(4\alpha t_m)^{1/2}$
t	y	$(t-t_m)/t_m$
$\tilde{H}(t)$	$F(y)$	$\tilde{H}(t)/H$
$T(x,t)$	$\theta(\zeta, y)$	$(T-T_0)/(T_m-T_0)$
$s(t)$	$\xi(y)$	$s/(4\alpha t_m)^{1/2}$
$\dot{s}(t)$	$\xi'(y)$	$\dot{s}[t_m/(4\alpha)]^{1/2}$
v	ν	$v[t_m/(4\alpha)]^{1/2}$

Table 2. Short and long time solutions of the embedding equations

	$0 < m < \infty,\ y \ll 1$	$m = \infty,\ y \ll 1$	$0 < m \leqslant \infty,\ y \gg 1$
$F(y)$	$\frac{-2(y)^{1/2}}{\pi}$ [1]	$\frac{y}{\pi}$ [3]	$\frac{4}{\pi^{1/2}} v \exp[4v(vy-a)]$ [3]
$\xi(y)$	$\frac{2}{3}\frac{m(y)^{3/2}}{\pi}$ [1]	$\frac{y}{2(\pi)^{1/2}}$ [2, 3]	$vy-a$ [2, 3]
		$a = \frac{1}{\pi^{1/2}} - v$ [2, 3]	

cal algorithm for larger values of y but presented no numerical results. The algorithm proposed in [1] is unstable, some of the approximations used in it are incorrect, and no useful numerical information can be obtained from it. This article gives a stable numerical method for finding $\xi(y)$ and $F(y)$, considers the case $m = \infty$, and includes long time solutions for all values of m.

By letting

$$I(y,\zeta) = 4(y+1)i^2\,\mathrm{erfc}\left[\frac{\zeta}{(y+1)^{1/2}}\right] + \int_0^y F(y_1)\,\mathrm{erfc}\left[\frac{\zeta}{(y-y_1)^{1/2}}\right]\mathrm{d}y_1 \quad (2)$$

($i^2\,\mathrm{erfc}(x)$ and $\mathrm{erfc}(x)$ are error functions defined in [4]), one can obtain the following equations for $\xi(y)$ and $F(y)$:

$$\xi(y) = v\{y+1-I[y,\xi(y)]\} \quad (3)$$

from an energy balance on the receding material, and

$$\left[\frac{\partial I(y,\zeta)}{\partial \zeta}\right]_{\zeta=\xi(y)} = -\frac{4}{\pi^{1/2}} \quad (4)$$

from $T[s(t),t] = T_m$. Boley [1] used d/dy of (3) instead of (3). After $F(y)$ and $\xi(y)$ have been found, they can be used to find the dimensionless temperature $\theta(\zeta, y)$ from

$$\theta(\zeta,y) = -\frac{\pi^{1/2}}{4}\frac{\partial I(y,\zeta)}{\partial \zeta}. \quad (5)$$

The formulation of the numerical algorithm given in [3] is based on making use of several short and long time (i.e. $y \ll 1$ and $y \gg 1$) solutions of (3) and (4). Table 2 summarizes the cases considered and indicates whether they were obtained in [1], [2] or [3]. For the case $m = 0$ ($L = \infty$), [3] shows that

$$F(y) = -\frac{2}{\pi}\tan^{-1}(y)^{1/2} \quad (6)$$

$$\frac{\xi(y)}{v} = \frac{2}{\pi}\left[(y+1)\tan^{-1}(y)^{1/2} - y^{1/2}\right] \quad (7)$$

[3] also shows that for fixed values of y, (6) and (7) give the minimum values of $F(y)$ and $\xi(y)/v$ and that both functions increase as m increases. From general considerations of (3) and (4) and the solutions given in Table 2, [3] also shows that: for $0 < m < \infty$, small time values of $F(y)$ are negative, long time values are positive and $F(y) = 0$ only for $y = 0$ and one other positive value; for $m = \infty$, $F(y)$ is never negative; and $\xi(y)/v$ monotonically increases with y for all values of m, approaching the terminal solution from above.

Boley [1] shows that for $0 < m < \infty$ the small time solutions in Table 2 are necessary to establish a forward marching numerical scheme based on using

$$\xi_N = \xi_{N-1} + \xi'_{N-1}(y_N - y_{N-1}) \quad (8)$$

as a first approximation of $\xi(y_N)$ at the Nth time level in the calculation. [3] shows that this is also true for $m = \infty$. The terminal behavior of $F(y)$ given in Table 2 led to the use of a modified fictitious heat flux function $G(y)$:

$$G(y) = F(y)\exp(-4v^2y) \quad (9)$$

in [3]. This choice forces $G(\infty)$ to be a constant.

The interval of integration in (2) is subdivided from $[0, y_N]$ into $[0, y_0]$ and $[y_0, y_N]$, where $0 < y_0 \ll 1$ and y_0 is used as the starting time for the calculations. Using the short time solutions for $F(y)$, expressions for the $[0, y_0]$ integration are given in [3], where some errors in Boley's [1] corresponding expressions are also pointed out. The $[y_0, y_N]$ integration is replaced by a numerical quadrature representation:

$$\sum_{i=1}^{N} G_i A(y_i, y_{i-1}, y_N, \zeta). \quad (10)$$

$A(y_i, y_{i-1}, y_N, \zeta)$, given in [3], is obtained by exact integration in the interval $[y_{i-1}, y_i]$ assuming that $G(y) = G_i$ throughout the interval.

The numerical marching scheme begins at $N = 1$ and uses (8) to obtain a first approximation of ξ_N. ξ'_{N-1} is found from Table 2 for $N = 1$, and by differentiation of (3) with respect to y for $0 < m < \infty$ and (3) and (4) for $m = \infty$ when $N > 1$. Newton–Raphson [5] iterative refinements are then made by

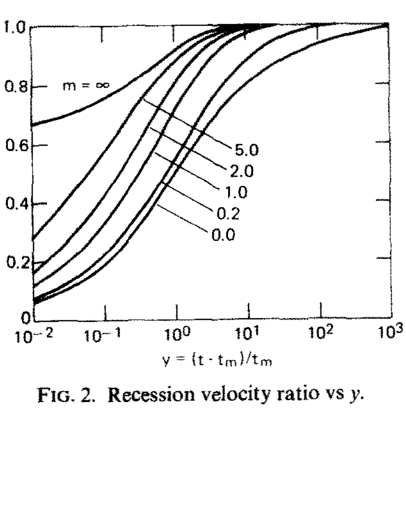

FIG. 2. Recession velocity ratio vs y.

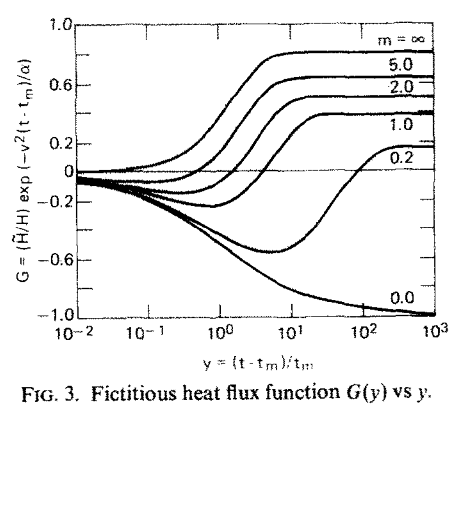

FIG. 3. Fictitious heat flux function $G(y)$ vs y.

finding G_N and ξ_N from numerical equivalents of (3) and (4). The steps in y were chosen to be small when y is small and to increase with N to preserve numerical stability, take advantage of the terminal behavior of the solutions, and maintain numerical accuracy. Further details of the mechanics of the numerical solution are given in [3].

Figures 2 and 3 show numerical results obtained for $\xi'(y)/v$ and $G(y)$. Additional plots of $\xi(y)/v$ and $\theta(0.5, y)$ are given in [3]. In all cases, the trends of the results agree with the qualitative expectations mentioned earlier, and the initial and terminal behavior agrees with the relations in Table 2. Calculations for different steps in y also agree well with each other. The curves in Fig. 2 agree with the corresponding Landau [2] curves when $y \geqslant 0(1)$ and $m < \infty$. [3] shows that when $y \ll 1$, the finite difference results obtained in [2] will overestimate $\xi'(y)$ by about 15% even for zero finite difference mesh size regardless of the value of $m < \infty$. Landau's [2] curve for $m = \infty$ is apparently misplotted because his plotted value of $\xi'(0)/v$ is incorrect.

A rapid and accurate numerical method for solving Boley's embedding equations applied to the LMP has been established. With only slight alterations, the algorithm is applicable to problems with time varying applied heat fluxes [3]. Landau's [2] finite difference solutions of the LMP have been used over the years to judge new techniques of analysis of melting problems. These include integral methods such as those proposed by Goodman [6] and Zien [7], and finite difference techniques such as those proposed by the author [3, 8, 9]. The primary value of the work presented here is that it provides a way to obtain accurate numerical values of the desired LMP parameters rapidly and economically for judging new approximate analytical and numerical methods of solving melting problems.

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